

MAS317 Linear Algebra II, 2007

Solutions to Coursework 5.

$$1. (a, b, c) \in U \text{ iff } \begin{matrix} a + b - 3c = 0 \\ a + 2b - 3c = 0 \end{matrix} \text{ iff } \begin{matrix} a + b - 3c = 0 \\ b = 0 \end{matrix}.$$

The non-leading variable is c . So a basis for U is $(3, 0, 1)$. [5 marks]

$(a, b, c) \in W$ iff $a - 3b - 5c = 0$. Non-leading variables are b, c and so take

$b = 1, c = 0$ giving $a = 3$ so vector $(3, 1, 0)$, and

$c = 1, b = 0$ giving $a = 5$ so vector $(5, 0, 1)$,

So a basis for W is $\{(3, 1, 0), (5, 0, 1)\}$. [5 marks]

(a) $U + W$ is spanned by $(3, 0, 1), (3, 1, 0), (5, 0, 1)$. [2 marks]

(b) This is a basis for $U + W$ as these 3 vectors are linearly independent since

$$\begin{pmatrix} 3 & 0 & 1 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - (5/3)R_1]{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2/3 \end{pmatrix}. \quad [5 \text{ marks}]$$

(c) By the Dimension Formula,

$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W) \geq 2 + 1 - 3 = 0$ So $U \cap W = \{0\}$ and the sum $U + W$ is direct. [3 marks]

$$2. (i) U = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 - 2x_2 + x_4 = 0, \text{ and } x_3 - 2x_4 = 0\}.$$

This is in echelon form and the non-leading variables are x_2, x_4, x_5 .

So a basis for U is given by finding the 3 solutions in which one of these is 1 and the others are 0. So a basis is

$(2, 1, 0, 0, 0), (-1, 0, 2, 1, 0), (0, 0, 0, 0, 1)$. [5 marks]

$$W = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 - 2x_2 + x_3 - x_4 = 0, \text{ and } x_2 + x_3 + x_5 = 0\}.$$

This is in echelon form and the non-leading variables are x_3, x_4, x_5 .

So a basis for W is given by finding the 3 solutions in which one of these is 1 and the others are 0. So a basis is

$(-3, -1, 1, 0, 0), (1, 0, 0, 1, 0), (-2, -1, 0, 0, 1)$. [5 marks]

U and W both have dimension 3. [2 marks]

(ii) $U \cap W = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 - 2x_2 + x_4 = 0, \text{ and } x_3 - 2x_4 = 0, \text{ and } x_1 - 2x_2 + x_3 - x_4 = 0 \text{ and } x_2 + x_3 + x_5 = 0\}$. **[1 marks]**

$= \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 - 2x_2 + x_4 = 0, \text{ and } x_2 + x_3 + x_5 = 0, \text{ and } x_3 - 2x_4 = 0\}$. as subtracting the first and second equations from the third gives $0 = 0$.

This is in echelon form and the non-leading variables are x_4, x_5 . So a basis for $U \cap W$ is given by finding the 2 solutions in which one of these is 1 and the other is 0. So a basis is

$(-5, -2, 2, 1, 0), (-2, -1, 0, 0, 1)$. **[8 marks]**

So the dimension of $U \cap W$ is 2 and the dimension of $U + W$ is $3 + 3 - 2 = 4$ by the Dimension Formula. **[1+3 marks]**

(iii) To use the Steinitz Exchange Lemma to find a basis for U consisting of your basis for $U \cap W$ and one or more vectors from your basis for U in (i) the method is: *take the basis for U , then and put in a vector from the basis for $U \cap W$ and remove a suitable vector from the original basis, then repeat this. (Exchange these two vectors.)*

A basis for U is $(2, 1, 0, 0, 0), (-1, 0, 2, 1, 0), (0, 0, 0, 0, 1)$. Now we want to exchange $(-2, -1, 0, 0, 1)$ for one of these three vectors.

Now $(-2, -1, 0, 0, 1) = -(2, 1, 0, 0, 0) + (0, 0, 0, 0, 1)$ so can exchange it for one of these two vectors, eg for $(0, 0, 0, 0, 1)$. **[4 marks]**

The new basis for U is $(-2, -1, 0, 0, 1), (2, 1, 0, 0, 0), (-1, 0, 2, 1, 0)$.

Now we want to exchange $(-5, -2, 2, 1, 0)$ for one of these three vectors (but not the first one). Now

$(-5, -2, 2, 1, 0) = -2(2, 1, 0, 0, 0) + (-1, 0, 2, 1, 0)$, so leave out one of these two.

So such a basis for U is $(-5, -2, 2, 1, 0), (-2, -1, 0, 0, 1), (2, 1, 0, 0, 0)$.

[4 marks]

A basis for W is $(-3, -1, 1, 0, 0), (1, 0, 0, 1, 0), (-2, -1, 0, 0, 1)$. Now

$(-2, -1, 0, 0, 1)$ is already in the basis for W . Now we want to

exchange $(-5, -2, 2, 1, 0)$ for one of the other vectors. Now

$(-5, -2, 2, 1, 0) = 2(-3, -1, 1, 0, 0) + (1, 0, 0, 1, 0)$, so leave out one of these two.

So such a basis for W is

$(-5, -2, 2, 1, 0), (-2, -1, 0, 0, 1), (-3, -1, 1, 0, 0)$. [5 marks]

(iv) A basis for $U + W$ containing bases for U and for W intersecting in a basis for $U \cap W$ is the union of the two bases found in (iii), so

$(-5, -2, 2, 1, 0), (-2, -1, 0, 0, 1), (2, 1, 0, 0, 0), (-3, -1, 1, 0, 0)$.

[2 marks]

3. By the Dimension Formula,

$\dim(V) \geq \dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$, so

$4 \geq 5 - \dim(U \cap W)$, so $\dim(U \cap W) \geq 1$. [2+3 marks]

$U \cap W \subseteq W$, so $\dim(U \cap W) \leq 2$.

[2 marks]

Hence $\dim(U \cap W) = 1$ or 2 .

[3 marks]

4. (i) If $V = U \oplus W_1$ and $V = U \oplus W_2$ then by the Dimension Formula

$\dim W_i = \dim V - \dim U - \dim(U \cap W_i) = \dim V - \dim U$

for $i = 1, 2$.

[5 marks]

If $\dim U < (\dim V)/2$ then $\dim W_i = \dim V - \dim U > (\dim V)/2$.

Now by the Dimension Formula

$\dim V \geq \dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) >$

$\dim V - \dim(W_1 \cap W_2)$, and so $W_1 \cap W_2 \neq \{0\}$.

[10 marks]

(ii) In $\mathbb{R}^4$, let $U = \langle (1, 1, -1, 0), (3, 3, 0, -4) \rangle$.

To find a subspace W_1 of $\mathbb{R}^4$ such that $\mathbb{R}^4 = U \oplus W_1$ by (4.5) we need to extend the basis for U to a basis for $\mathbb{R}^4$ and take W_1 to be spanned by the two extra vectors.

Putting the vectors in U into a matrix and doing $R_2 \rightarrow R_2 - 3R_1$ gives the matrix

$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & -4 \end{pmatrix}$. We want to put in two extra rows so that the new

matrix has row rank 4. For example $(0, 1, 0, 0)$ and $(0, 0, 0, 1)$ will do.

So one answer is

$W_1 = \langle (0, 1, 0, 0), (0, 0, 0, 1) \rangle = \{(0, a, 0, b) \mid a, b \in \mathbb{R}\}$.

[5 marks]

Then to find a subspace W_2 of $\mathbb{R}^4$ such that $\mathbb{R}^4 = U \oplus W_2$ and

$W_1 \cap W_2 = \{0\}$ we want to solve the same problem finding a

different choice of two vectors to span W_2 so that the new matrix has row rank 4, AND we need to arrange our choice such that $W_1 \cap W_2 = \{\underline{0}\}$. (Note: It is no good adding $(1, 1, -1, 0)$ to both of $(0, 1, 0, 0), (0, 0, 0, 1)$, as then the difference of the two new vectors is in W_1 .) There are many ways to solve this.

Eg. Put $w_1 = (1, 1, -1, 0) + (0, 1, 0, 0)$ and

$w_2 = (0, 0, 3, -4) + (0, 0, 0, 1)$, then the matrix $\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & -4 \\ 1 & 2 & -1 & 0 \\ 0 & 0 & 3 & -3 \end{pmatrix}$ is

row equivalent to $\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and so has row rank 4. This

gives

$$W_2 = \langle (1, 2, -1, 0), (0, 0, 3, -3) \rangle = \{(c, 2c, 3d - c, -3d) \mid c, d \in \mathbb{R}\}.$$

Clearly if $(c, 2c, 3d - c, -3d) \in W_1$ then $c = d = 0$, and so

$$W_1 \cap W_2 = \{\underline{0}\}.$$

[10 marks]